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Properties of unilevel block circulants

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ABSTRACT

Let $\mathscr{A}=\{A_0,A_1,\ldots,A_{k-1}\}\subset\mathbb{C}^{d_1\times d_2}$, $\zeta=e^{-2\pi i/k}$, $F_\ell=\sum_{m=0}^{k-1}\zeta^{\ell m}A_m$, $0\leqslant\ell\leqslant k-1$, and $\mathscr{F}_A=\bigoplus_{\ell=0}^{k-1}F_\ell$. All operations in indices are modulo k. It is well known that if $d_1=d_2=1$ then $[A_{s-r}]_{r,s=0}^{k-1}=\Phi\mathscr{F}_A\Phi^*$, where $\Phi=\frac{1}{\sqrt{k}}\left[\zeta^{\ell m}\right]_{\ell,m=0}^{k-1}$. However, to our knowledge it has not been emphasized that $\mathscr{F}_A$ plays a fundamental role in connection with all the matrices $[A_{s-\alpha r}]_{r,s=0}^{k-1}$, $0\leqslant\alpha\leqslant k-1$, with d_1,d_2 arbitrary. We begin by adapting a theorem of Ablow and Jenner with $d_1=d_2=1$ to the case where d_1 and d_2 are arbitrary. We show that $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if $A=U_\alpha\mathscr{F}_AP^*$ where U_α and P are related to Φ, P is unitary, and U_α is invertible (in fact, unitary) if and only if $\gcd(\alpha,k)=1$, in which case we say that A is a proper circulant. We prove the following for proper circulants $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$: (i) $A^\dagger=[B_{r-\alpha s}]_{r,s=0}^{k-1}$ with $B_m=\frac{1}{k}\sum_{\ell=0}^{k-1}\zeta^{\ell m}F_\ell^\dagger$, $0\leqslant m\leqslant k-1$. (ii) Solving $Az=w$ reduces to solving $F_\ell u_\ell=v_{\alpha\ell}$, $0\leqslant\ell\leqslant k-1$, where $v_0,v_1,\ldots,v_{k-1}$ depend only on w. (iii) A singular value decomposition of A can be obtained from singular value decompositions of $F_0,F_1,\ldots,F_{k-1}$. (iv) The least squares problem for A reduces to independent least squares problems for $F_0,F_1,\ldots,F_{k-1}$. (v) If $d_1=d_2=d$, the eigenvalues of $[A_{s-r}]_{r,s=0}^{k-1}$ are the eigenvalues of $F_0,F_1,\ldots,F_{k-1}$, and the corresponding eigenvectors of A are easily obtainable from those of $F_0,F_1,\ldots,F_{k-1}$. (vi) If $d_1=d_2=d$ and $\alpha>1$ then the eigenvalue problem for $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ reduces to eigenvalue problems for $d\times d$ matrices related to $F_0,F_1,\ldots,F_{k-1}$ in a manner depending upon α.

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1. Introduction

Throughout this paper $k \geqslant 2, d_1, d_2 \geqslant 1$ are integers, $\alpha \in \{0, 1, \ldots, k-1\}$, and

$$\mathbb{C}^{k:d_1\times d_2} = \{C = [C_{rs}]_{r,s=0}^{k-1} \,\Big|\, C_{rs} \in \mathbb{C}^{d_1\times d_2}, 0 \leqslant r, s \leqslant k-1\}.$$

All arithmetic operations in indices are modulo k.

We call $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\times d_2}$ an α-circulant. We say that A is a proper α-circulant, or simply a proper circulant, if $\gcd(\alpha, k) = 1$. We will say that A is a standard α-circulant if $d_1 = d_2 = 1$ and denote it by $A = [a_{s-ar}]_{r,s=0}^{k-1}$. Of course, there is already a vast literature on standard α-circulants. Matrices of the form

$$A = [A_{rs}]_{r,s=0}^{k-1} \quad \text{where} \quad A_{rs} = \begin{cases} A_{s-r}, & 0 \leqslant r \leqslant s \leqslant k-1, \\ kA_{s-r}, & 0 \leqslant s < r \leqslant k-1, \end{cases}$$

are also called k-circulants; see e.g., [4]. We will not consider them.

We call $[B_{r-\alpha s}]_{r,s=0}^{k-1}$ an α-cocirculant, again proper if $\gcd(\alpha, k) = 1$. This eliminates awkward terminology such as "the conjugate transpose of the Moore–Penrose inverse of an α-circulant matrix is an α-circulant". The Moore–Penrose inverse of an α-circulant is an α-cocirculant (Theorem 4).

Remark 1. Obviously, B is an α-cocirculant if and only if B^* is an α-circulant. Therefore any result concerning α-circulants can be applied to B^* to obtain a result concerning B.

Remark 2. A proper α-circulant $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ is also a β-cocirculant where $\alpha\beta \equiv 1 \pmod{k}$, since

$$A_{s-\alpha r} = A_{\alpha\beta s-\alpha r} = A_{-\alpha(r-\beta s)} = B_{r-\beta s}$$

with $B_m = A_{-\alpha m}, 0 \leqslant m \leqslant k-1$. Similarly, a proper β-cocirculant $B = [B_{r-\beta s}]_{r,s=0}^{k-1}$ is also an α-circulant, since

$$B_{r-\beta s} = B_{\alpha\beta r-\beta s} = B_{-\beta(s-\alpha r)} = C_{s-\alpha r}$$

with $C_m = B_{-\beta m}, 0 \leqslant m \leqslant k-1$.

Henceforth $\zeta = e^{-2\pi i/k}$,

$$E = [\delta_{\ell,m-1}]_{\ell,m=0}^{k-1}, \quad \text{and} \quad \Phi = \frac{1}{\sqrt{k}}[\zeta^{\ell m}]_{\ell,m=0}^{k-1} = \begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_{k-1} \end{bmatrix} \tag{1}$$

(the Fourier matrix), with

$$\phi_m = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^m \\ \zeta^{2m} \\ \vdots \\ \zeta^{(k-1)m} \end{bmatrix}, \quad 0 \leqslant m \leqslant k-1. \tag{2}$$

It is straightforward to verify that if indices are reduced modulo k then

$$E^p \left([g_{\ell m}]_{\ell,m=0}^{k-1} \right) E^{-q} = [g_{\ell+p,m+q}]_{\ell,m=0}^{k-1}. \tag{3}$$

Setting $p = 1$ and $q = 0$ and invoking (1) yields

$$E\Phi = \frac{1}{\sqrt{k}}[\zeta^{(\ell+1)m)}]_{\ell,m=0}^{k-1} = \Phi D \quad \text{with} \quad D = \operatorname{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{k-1}). \tag{4}$$

Therefore $E = \Phi D \Phi^*$.

The discrete Fourier transform (DFT) of $\{A_0, A_1, \ldots, A_{k-1}\} \subset \mathbb{C}^{d_1\times d_2}$ is $\{F_0, F_1, \ldots, F_{k-1}\}$ where

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m \in \mathbb{C}^{d_1\times d_2}, \quad 0 \leqslant \ell \leqslant k-1. \tag{5}$$

Since $\Phi^{-1} = \Phi^*$,

$$A_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} F_\ell, \quad 0 \leqslant m \leqslant k-1. \tag{6}$$

We denote

$$\mathscr{F}_A = \bigoplus_{\ell=0}^{k-1} F_\ell \in \mathbb{C}^{k:d_1 \times d_2}. \tag{7}$$

For standard circulants (5)–(7) reduce to

$$f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m}, \quad a_m = \frac{1}{k}\sum_{\ell=0}^{k-1} f_\ell \zeta^{-\ell m} \quad \text{and} \quad \mathscr{F}_A = \operatorname{diag}(f_0, f_1, \ldots, f_{k-1}).$$

It is well known (see, e.g. [7]) that a standard 1-circulant $A = [a_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times k}$ can be written as

$$A = \Phi \mathscr{F}_A \Phi^* = \sum_{\ell=0}^{k-1} f_\ell \phi_\ell \phi_\ell^*.$$

However, to our knowledge it has not been emphasized that $\mathscr{F}_A$ plays a fundamental role in connection with all the standard circulants $[a_{s-\alpha r}]_{r,s=0}^{k-1}$. (See Remark 3.)

In Section 2 we reformulate a result of Ablow and Jenner [1, Theorem 2.1] for standard α-circulants to characterize α-circulants in $\mathbb{C}^{k:d_1 \times d_2}$. We give a different characterization in Section 3: $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if $A = U_\alpha \mathscr{F}_A P^*$, where U_α and P are related to the Fourier matrix, P is unitary, and U_α is invertible (in fact, unitary) if and only if $\gcd(\alpha, k) = 1$.

Since $\mathscr{F}_A$ is independent of α, some computational results concerning $\mathscr{F}_A$ apply simultaneously to all the proper α-circulants $[A_{s-\alpha r}]_{r,s=0}^{k-1}$. For example, in Section 4 we show that

$$A^\dagger = [B_{r-\alpha s}]_{r,s=0}^{k-1}, \quad \text{where } B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^\dagger, \quad 0 \leqslant m \leqslant k-1.$$

We also prove the following for proper α-circulants: (i) Solving $Az = w$ reduces to solving $F_\ell u_\ell = v_{\alpha\ell}, 0 \leqslant \ell \leqslant k-1$, where $v_0, v_1, \ldots, v_{k-1}$ depend only on w. (ii) A singular value decomposition of A can be obtained from singular value decompositions of $F_0, F_1, \ldots, F_{k-1}$. (iii) The least squares problem for A reduces to independent least squares problems for $F_0, F_1, \ldots, F_{k-1}$. (iv) If $d_1 = d_2 = d$, the eigenvalues of $[A_{s-r}]_{r,s=0}^{k-1}$ are the eigenvalues of $F_0, F_1, \ldots, F_{k-1}$, and the corresponding eigenvectors of A are easily obtainable from those of $F_0, F_1, \ldots, F_{k-1}$. (v) If $d_1 = d_2 = d$ and $\alpha > 1$, the eigenvalue problem for $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ reduces to eigenvalue problems for $d \times d$ matrices related to $F_0, F_1, \ldots, F_{k-1}$ in a manner depending upon α.

Block circulant 1-matrices $[A_{s-r}]_{r,s=0}^{k-1}$ have applications in preconditioning of block Toeplitz matrices; see, e.g. [8,9].

2. The Ablow–Jenner theorem revisited

Recall that E and Φ are defined in (1) and (2). Let

$$R = E \otimes I_{d_1}, \quad P_m = \phi_m \otimes I_{d_1}, \quad 0 \leqslant m \leqslant k-1, \tag{8}$$

$$S = E \otimes I_{d_2}, \quad Q_m = \phi_m \otimes I_{d_2}, \quad 0 \leqslant m \leqslant k-1, \tag{9}$$

$$P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix}, \tag{10}$$

and

$$U_\alpha = \begin{bmatrix} P_0 & P_\alpha & P_{2\alpha} & \cdots & P_{(k-1)\alpha} \end{bmatrix}. \tag{11}$$

Since

$$P_\ell^* P_m = \delta_{\ell m} I_{d_1} \quad \text{and} \quad Q_\ell^* Q_m = \delta_{\ell m} I_{d_2}, \quad 0 \leqslant \ell, m \leqslant k-1,$$

P and Q are unitary, while U_α is unitary if $\gcd(\alpha, k) = 1$ and $p = k/q$; however, if $\gcd(\alpha, k) = q > 1$ and $p = k/q$, then

$$U_\alpha = [\underbrace{P_0 P_\alpha \cdots P_{(p-1)\alpha} \cdots P_0 P_\alpha \cdots P_{(p-1)\alpha}}_{q}]$$

(i.e., the first p block columns are repeated q times) is not invertible. From (4) and (8)–(11),

$$RP_\ell = \zeta^\ell P_\ell \quad \text{and} \quad SQ_\ell = \zeta^\ell Q_\ell, \quad 0 \leqslant \ell \leqslant k-1. \tag{12}$$

Ablow and Jenner [1, Theorem 2.1] $A \in \mathbb{C}^{k \times k}$ is a standard α-circulant if and only if $EAE^{-\alpha} = A$. We need the following adaptation of this result.

Theorem 1. *If* $A = [G_{rs}]_{r,s=0}^{k-1}$ with $G_{rs} \in \mathbb{C}^{d_1 \times d_2}$, *then* $RAS^{-\alpha} = A$ *if and only if* A *is an* α*-circulant; more precisely, if and only if*

$$G_{rs} = A_{s-\alpha r}, \quad 0 \leqslant r,\ s \leqslant k-1, \tag{13}$$

with

$$A_s = G_{0s}, \quad 0 \leqslant s \leqslant k-1. \tag{14}$$

Proof. From (3), (8), and (9), $RAS^{-\alpha} = [G_{r+1,s+\alpha}]_{r,s=0}^{k-1}$. Therefore we must show that (13) is equivalent to

$$G_{r+1,s+\alpha} = G_{rs}, \quad 0 \leqslant r, s \leqslant k-1. \tag{15}$$

If (13) holds, then

$$G_{r+1,s+\alpha} = A_{(s+\alpha)-(r+1)\alpha} = A_{s-\alpha r} = G_{rs}, \quad 0 \leqslant r,\ s \leqslant k-1.$$

For the converse we must show that (14) and (15) imply (13). We prove this by finite induction on r. From (14),

$$G_{rs} = A_{s-\alpha r}, \quad 0 \leqslant s \leqslant k-1, \tag{16}$$

if $r = 0$. Suppose (16) is true for some $r \in \{0, \ldots, k-2\}$. Replacing s by $s - \alpha$ in (15) and (16) yields

$$G_{r+1,s} = G_{r,s-\alpha}, \quad 0 \leqslant r,\ s \leqslant k-1,$$

and

$$G_{r,s-\alpha} = A_{s-\alpha(r+1)}, \quad 0 \leqslant s \leqslant k-1.$$

Therefore

$$G_{r+1,s} = A_{s-\alpha(r+1)}, \quad 0 \leqslant s \leqslant k-1,$$

which completes the induction. □

Theorem 1 with $A = B^*$ yields the following corollary.

Corollary 1. *If* $B \in \mathbb{C}^{k:d_2 \times d_1}$ *then* B *is an* α*-cocirculant if and only if* $S^\alpha BR^{-1} = B$.

The following corollary extends [10, Corollary 1].

Corollary 2. (i) *If* $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}$ *and* $B = [B_{r-\alpha s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_2 \times d_1}$, *then* $AB = [C_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_1}$ *with* $C_m = \sum_{\ell=0}^{k-1} A_\ell B_{\ell-\alpha m}, 0 \leqslant m \leqslant k-1$.

(ii) *If* $\gcd(\alpha,k)=1$ *and* $\alpha\beta\equiv 1 \pmod{k}$, *then* $BA=[D_{s-r}]_{r,s=0}^{k-1}\in\mathbb{C}^{k:d_2\times d_2}$ *with*

$$D_m=\sum_{\ell=0}^{k-1}B_\ell A_{m+\ell},\quad 0\leqslant m\leqslant k-1. \tag{17}$$

Proof. (i) From Theorem 1 and Corollary 1, $A=RAS^{-\alpha}$ and $B=S^\alpha BR^{-1}$. Therefore $AB=RABR^{-1}$, so Theorem 1 with $R=S$ implies that AB is a 1-circulant. The stated formula for $C_0,C_1,\ldots,C_{k-1}$ can be obtained by computing first block row entries of AB.

(ii) Also, $BA=S^\alpha BAS^{-\alpha}$. Applying this β times yields $BA=SBAS^{-1}$, so Theorem 1 with $R=S$ implies that BA is a 1-circulant. Computing the first block row entries of BA yields $D_m=\sum_{\ell=0}^{k-1}B_{-\alpha\ell}A_{m-\alpha\ell}$ and replacing ℓ by $-\beta\ell$ yields (17). □

Theorem 2. *If*

$$A=[A_{s-\alpha_1 r}]_{r,s=0}^{k-1}\in\mathbb{C}^{k:d_1\times d_2} \tag{18}$$

and

$$B=[B_{s-\alpha_2 r}]_{r,s=0}^{k-1}\in\mathbb{C}^{k:d_2\otimes d_3}, \tag{19}$$

then

$$AB=[C_{s-\alpha_1\alpha_2 r}]_{r,s=0}^{k-1}\in\mathbb{C}^{k:d_1\otimes d_3}, \tag{20}$$

with

$$C_m=\sum_{\ell=0}^{k-1}A_\ell B_{m-\alpha_2\ell},\quad 0\leqslant m\leqslant k-1. \tag{21}$$

Proof. Let $R=E\otimes I_{d_1}$, $S=E\otimes I_{d_2}$, and $T=E\otimes I_{d_3}$. From (18), (19), and Theorem 1,

(a) $A=RAS^{-\alpha_1}$ and (b) $B=SBT^{-\alpha_2}$.

Applying (b) α_1 times yields $B=S^{\alpha_1}BT^{-\alpha_1\alpha_2}$. From this and (a), $RABT^{-\alpha_1\alpha_2}=AB$. Now Theorem 1 implies (20), with (21) obtained by computing the entries in the first block row of AB. □

Theorem 2 generalizes [1, Theorem 3.1]; namely, the product of a standard α-circulant and a standard β-circulant is an $\alpha\beta$-circulant. However, [1] does not include (21).

3. A DFT characterization of α-circulants

Theorem 3. *A matrix* $A\in\mathbb{C}^{k:d_1\times d_2}$ *is an* α*-circulant* $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ *if and only if it can be written as*

$$A=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell Q_\ell^*=U_\alpha\mathscr{F}_AQ^*, \tag{22}$$

where $\{F_0,F_1,\ldots,F_{k-1}\}$ *and* $\{A_0,A_1,\ldots,A_{k-1}\}$ *are related as in* (5) *and* (6) *and* P, Q, *and* U_α *are as in* (8)–(11).

Proof. Eqs. (7)–(11) imply the second equality in (22). Therefore we need only justify the first. Suppose $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ and define $F_0,F_1,\ldots,F_{k-1}$ by (5). From (6),

$$A_{s-\alpha r}=\frac{1}{k}\sum_{\ell=0}^{k-1}\zeta^{-\ell(s-\alpha r)}F_\ell,\quad 0\leqslant r,s\leqslant k-1,$$

so (8)–(11) imply that

$$A=\frac{1}{k}\sum_{\ell=0}^{k-1}\begin{bmatrix}1\otimes I_{d_1}\\ \zeta^{\alpha\ell}\otimes I_{d_1}\\ \vdots\\ \zeta^{(k-1)\alpha\ell}\otimes I_{d_1}\end{bmatrix}F_\ell\begin{bmatrix}1\otimes I_{d_2}\\ \zeta^{\ell}\otimes I_{d_2}\\ \vdots\\ \zeta^{(k-1)\ell}\otimes I_{d_2}\end{bmatrix}^H=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell Q_\ell^*.$$

For the converse, if (22) holds then (12) implies that

$$RAS^{-\alpha}=\sum_{\ell=0}^{k-1}(RP_{\alpha\ell})F_\ell(S^\alpha Q_\ell)^*=\sum_{\ell=0}^{k-1}(\zeta^{\alpha\ell}P_{\alpha\ell})F_\ell(\zeta^{-\alpha\ell}Q_\ell^*)=A.$$

Therefore A is an α-circulant, by Theorem 1; hence, $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ with $A_0,A_1,\dots,A_{k-1}$ as in (6). $\square$

Remark 3. Theorem 3 implies that $A\in\mathbb{C}^{k\times k}$ is a standard α-circulant $[a_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if

$$A=\Phi_\alpha\mathscr{F}_A\Phi^*=\sum_{\ell=0}^{k-1}f_\ell\phi_{\alpha\ell}\phi_\ell^*,$$

where Φ is as in (1), $\Phi_\alpha=[\phi_0\quad\phi_\alpha\quad\cdots\quad\phi_{(k-1)\alpha}]$, and

$$f_\ell=\sum_{m=0}^{k-1}a_m\zeta^{\ell m},\quad 0\leqslant\ell\leqslant k-1.$$

Corollary 3. *A matrix $B\in\mathbb{C}^{kd_2\times d_1}$ is an α-cocirculant if and only if it can be written as $B=\sum_{\ell=0}^{k-1}Q_\ell G_\ell P_{\alpha\ell}^*$, where*

$$G_\ell=\sum_{m=0}^{k-1}\zeta^{-\ell m}B_m,\quad 0\leqslant\ell\leqslant k-1\quad\text{and}\quad B_m=\frac{1}{k}\sum_{\ell=0}^{k-1}\zeta^{\ell m}G_\ell,\quad 0\leqslant m\leqslant k-1.$$

Proof. Apply Theorem 3 to B^*. $\square$

It is well known that standard 1-circulants commute. The following corollary extends this.

Corollary 4. *Suppose $d_1=d_2$, $\gcd(\alpha,k)=1$, and $\alpha\beta\equiv 1\pmod k$. Let $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$, $B=[B_{s-\beta r}]_{r,s=0}^{k-1}$,*

$$F_\ell=\sum_{m=0}^{k-1}\zeta^{\ell m}A_m\quad\textit{and}\quad G_\ell=\sum_{m=0}^{k-1}\zeta^{\ell m}B_m.$$

Then $AB=BA$ if and only if $F_{\beta\ell}G_\ell=G_{\alpha\ell}F_\ell$, $0\leqslant\ell\leqslant k-1$.

Proof. Since $\gcd(\alpha,k)=\gcd(\beta,k)=1$, we may change summation indices $\ell\to\alpha\ell$ and $\ell\to\beta\ell$. Therefore, from Theorem 3 with $Q=P$,

$$A=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell P_\ell^*=\sum_{\ell=0}^{k-1}P_\ell F_{\beta\ell}P_{\beta\ell}^*,\quad B=\sum_{\ell=0}^{k-1}P_{\beta\ell}G_\ell P_\ell^*=\sum_{\ell=0}^{k-1}P_\ell G_{\alpha\ell}P_{\alpha\ell}^*,$$
$$AB=\sum_{\ell=0}^{k-1}P_\ell F_{\beta\ell}G_\ell P_\ell^*,\quad\text{and}\quad BA=\sum_{\ell=0}^{k-1}P_\ell G_{\alpha\ell}F_\ell P_\ell^*,$$

which implies the conclusion. $\square$

4. Moore–Penrose inversion and singular value decomposition

Recall that the Moore–Penrose inverse $X^\dagger$ of a matrix X is the unique matrix Y that satisfies the Penrose conditions

$$(XY)^* = XY, \quad (YX)^* = YX, \quad XYX = X \quad \text{and} \quad YXY = Y.$$

Theorem 4. *The Moore–Penrose inverse of an α-circulant is an α-cocirculant.*

Proof. From Theorem 1, if A is an α-circulant then $A = RAS^{-\alpha}$. Let $B = S^\alpha A^\dagger R^{-1}$. We will show that A and B satisfy the Penrose conditions:

$$\begin{aligned}
AB &= (RAS^{-\alpha})(S^\alpha A^\dagger R^{-1}) = RAA^\dagger R^* = R(AA^\dagger)^* R^* = (AB)^*,\\
BA &= (S^\alpha A^\dagger R^{-1})(RAS^{-\alpha}) = S^\alpha A^\dagger A(S^\alpha)^* = S^\alpha (A^\dagger A)^*(S^\alpha)^* = (BA)^*,\\
ABA &= (RAA^\dagger R^{-1})(RAS^{-\alpha}) = R(AA^\dagger A)S^{-\alpha} = RAS^{-\alpha} = A
\end{aligned}$$

and

$$BAB = (S^\alpha A^\dagger AS^{-\alpha})(S^\alpha A^\dagger R^{-1}) = S^\alpha (A^\dagger AA^\dagger)R^{-1} = S^\alpha A^\dagger R^{-1} = B.$$

Therefore $B = A^\dagger$ or, equivalently, $S^\alpha A^\dagger R^{-1} = A^\dagger$. Now Corollary 1 implies that $A^\dagger$ is an α-cocirculant. □

We can be more explicit if $\gcd(\alpha, k) = 1$.

Theorem 5. *The Moore–Penrose inverse of a proper α-circulant $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ is the α-cocirculant $B = [B_{r-\alpha s}]_{r,s=0}^{k-1}$, where*

$$B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^\dagger, \quad 0 \leqslant m \leqslant k-1, \tag{23}$$

with

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m, \quad 0 \leqslant \ell \leqslant k-1.$$

Proof. From Theorem 3, $A = U_\alpha \mathscr{F}_A Q^*$ where Q and U_α are unitary, the latter since $\gcd(\alpha, k) = 1$. We will first show that A and $B = Q\mathscr{F}_A^\dagger U_\alpha^*$ satisfy the Penrose conditions:

$$\begin{aligned}
AB &= (U_\alpha \mathscr{F}_A Q^*)(Q\mathscr{F}_A^\dagger U_\alpha^*) = U_\alpha \mathscr{F}_A \mathscr{F}_A^\dagger U_\alpha^* = U_\alpha (\mathscr{F}_A \mathscr{F}_A^\dagger)^* U_\alpha^* = (AB)^*,\\
BA &= (Q\mathscr{F}_A^\dagger U_\alpha^*)(U_\alpha \mathscr{F}_A Q^*) = Q\mathscr{F}_A^\dagger \mathscr{F}_A Q^* = Q(\mathscr{F}_A^\dagger \mathscr{F}_A)^* Q^* = (BA)^*,\\
ABA &= (U_\alpha \mathscr{F}_A \mathscr{F}_A^\dagger U_\alpha^*)(U_\alpha \mathscr{F}_A Q^*) = U_\alpha (\mathscr{F}_A \mathscr{F}_A^\dagger \mathscr{F}_A) Q^* = U_\alpha \mathscr{F}_A Q^* = A,
\end{aligned}$$

and

$$BAB = (Q\mathscr{F}_A^\dagger \mathscr{F}_A Q^*)(Q\mathscr{F}_A^\dagger U_\alpha^*) = Q(\mathscr{F}_A^\dagger \mathscr{F}_A \mathscr{F}_A^\dagger)U_\alpha^* = Q\mathscr{F}_A^\dagger U_\alpha^* = B.$$

Therefore

$$\begin{aligned}
A^\dagger = B &= \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger P_{\alpha\ell}^* = \sum_{\ell=0}^{k-1} (\phi_\ell \otimes I_{d_2}) F_\ell^\dagger (\phi_{\alpha\ell} \otimes I_{d_1})^*\\
&= \frac{1}{k}\left[\sum_{\ell=0}^{k-1} \zeta^{\ell(r-\alpha s)} F_\ell^\dagger\right]_{r,s=0}^{k-1} = [B_{r-\alpha s}]_{r,s=0}^{k-1},
\end{aligned}$$

from (8)–(11) and (23). □

Remark 4. Theorem 5 can also be proved by using (6) and (23) to express the entries of AB, BA, ABA, and BAB explicitly in terms of $F_0, F_1, \ldots, F_{k-1}$ and $F_0^\dagger, F_1^\dagger, \ldots, F_{k-1}^\dagger$, noting that

$$\sum_{\ell=0}^{k-1} \zeta^{\ell(r-s)} = \sum_{\ell=0}^{k-1} \zeta^{\alpha\ell(r-s)} = \delta_{rs}, \quad 0 \leqslant r, s \leqslant k-1,$$

the latter because $\gcd(\alpha, k) = 1$. However, this is tedious.

Remark 5. Theorem 5 extends a result of Davis [6]: If $A = [a_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k\times k}$ then $A^\dagger = \Phi \operatorname{diag}(a_0^\dagger, a_1^\dagger, \ldots, a_{k-1}^\dagger)\Phi^*$, where Φ is the Fourier matrix (1), $0^\dagger = 0$, and $a^\dagger = 1/a$ if $a \neq 0$.

Theorem 6. *Suppose* $\gcd(\alpha, k) = 1$ *and*

$$A = [A_{s-\alpha r}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^* = U_\alpha \mathscr{F}_A Q^*.$$

Let $F_\ell = \Omega_\ell \Sigma_\ell \Psi_\ell^*$ *be a singular value decomposition of* F_ℓ, $0 \leqslant \ell \leqslant k-1$, *and define*

$$M_\alpha = \begin{bmatrix} P_0\Omega_0 & P_\alpha\Omega_1 & \cdots & P_{(k-1)\alpha}\Omega_{k-1} \end{bmatrix}$$

and

$$N = \begin{bmatrix} Q_0\Psi_0 & Q_1\Psi_1 & \cdots & Q_{k-1}\Psi_{k-1} \end{bmatrix}.$$

Then

$$A = M_\alpha \left(\bigoplus_{\ell=0}^{k-1} \Sigma_\ell \right) N^*$$

is a singular value decomposition of A, *except that the singular values are not necessarily ordered*.

5. The least squares problem

Suppose $G \in \mathbb{C}^{d_1\times d_2}$ and consider the least squares problem for G: If $v \in \mathbb{C}^{d_1}$, find $u \in \mathbb{C}^{d_2}$ such that

$$\|Gu - v\| = \min_{\xi\in\mathbb{C}^{d_2}} \|G\xi - v\|, \tag{24}$$

where $\|\cdot\|$ is the 2-norm. This problem has a unique solution if and only if $\operatorname{rank}(G) = d_2$, in which case $u = (G^*G)^{-1}G^*v$. In any case, the optimal solution of (24) is the unique $u_0 \in \mathbb{C}^{d_2}$ of minimum norm that satisfies (24); thus, $u_0 = G^\dagger v$. The general solution of (24) is $u = u_0 + q$ where $Gq = 0$, and

$$\|Gu - v\| = \|(GG^\dagger - I)v\|$$

for all such u.

Now consider the following least squares problem: if $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\times d_2}$ with $\gcd(\alpha, k) = 1$ and $w \in \mathbb{C}^{kd_1}$, find $z \in \mathbb{C}^{kd_2}$ such that

$$\|Az - w\| = \min_{\xi\in\mathbb{C}^{kd_2}} \|A\xi - w\|.$$

We write

$$z = \sum_{\ell=0}^{k-1} Q_\ell u_\ell \quad \text{and} \quad w = \sum_{\ell=0}^{k-1} P_\ell v_\ell = \sum_{\ell=0}^{k-1} P_{\alpha\ell} v_{\alpha\ell}, \tag{25}$$

since substituting $\alpha\ell$ for ℓ is legitimate because $\gcd(\alpha,k)=1$. Since $A=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell Q_\ell^*$ and $Q_\ell^*Q_m=\delta_{\ell m}I_{kd_2}$,

$$Az-w=\sum_{\ell=0}^{k-1}P_{\alpha\ell}(F_\ell u_\ell-v_{\alpha\ell}).$$

Since $P_{\alpha\ell}^*P_{\alpha m}=\delta_{\ell m}I_{d_1}$ (because $\gcd(\alpha,k)=1$), it follows that

$$\|Az-w\|^2=\sum_{\ell=0}^{k-1}\|F_\ell u_\ell-v_{\alpha\ell}\|^2. \tag{26}$$

This implies the following theorem.

Theorem 7. *Suppose A is a proper α-circulant and let z and w be as in (25). Then*

$$\|Az-w\|=\min_{\xi\in\mathbb{C}^{kd_2}}\|A\xi-w\| \tag{27}$$

if and only if

$$\|F_\ell u_\ell-v_{\alpha\ell}\|=\min_{\psi_\ell\in C^{d_2}}\|F_\ell\psi_\ell-v_{\alpha\ell}\|,\quad 0\leqslant\ell\leqslant k-1.$$

Therefore (27) has a unique solution, given by

$$z=\sum_{\ell=0}^{k-1}Q_\ell(F_\ell^*F_\ell)^{-1}F_\ell^*v_{\alpha\ell},$$

if and only $\operatorname{rank}(F_\ell)=d_2, 0\leqslant\ell\leqslant k-1$. *In any case, the optimal solution of (27) is*

$$z_0=\sum_{\ell=0}^{k-1}Q_\ell F_\ell^\dagger v_{\alpha\ell}.$$

The general solution of (27) is $z=z_0+\sum_{\ell=0}^{k-1}Q_\ell u_\ell$, where $F_\ell u_\ell=0, 0\leqslant\ell\leqslant k-1$, and

$$\|Az-w\|^2=\sum_{\ell=0}^{k-1}\|(F_\ell F_\ell^\dagger-I_{d_1})v_{\alpha\ell}\|^2$$

for all such z.

6. The case where $d_1=d_2$

Throughout this section $d_1=d_2=d$ and $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ is a proper circulant. Then (26) implies the following theorem, which reduces the problem of solving the $kd\times kd$ system $Az=w$ to solving k independent $d\times d$ systems.

Theorem 8. *If A is a proper α-circulant, $z=\sum_{\ell=0}^{k-1}P_\ell u_\ell$, and $w=\sum_{\ell=0}^{k-1}P_\ell v_\ell$, then $Az=w$ if and only if*

$$F_\ell u_\ell=v_{\alpha\ell},\quad 0\leqslant\ell\leqslant k-1.$$

This and Theorem 5 imply the following theorem.

Theorem 9. *A proper α-circulant*

$$A=[A_{s-\alpha r}]_{r,s=0}^{k-1}=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell P_\ell^* \tag{28}$$

is invertible if and only $F_0, F_1, \ldots, F_{k-1}$ are all invertible. In this case

$$A^{-1} = [B_{r-\alpha s}]_{r,s=0}^{k-1} \quad \text{with } B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^{-1}, \quad 0 \leqslant m \leqslant k-1,$$

and the solution of $Az = w$ is $z = \sum_{\ell=0}^{k-1} P_\ell F_\ell^{-1} v_{\alpha\ell}$.

Remark 6. Theorem 9 and Remark 2 extend [5, Theorem 1]: the inverse of a standard nonsingular α-circulant is a β-circulant, where $\alpha\beta \equiv 1 \pmod{k}$.

Theorem 10. *Suppose A is a proper α-circulant as in (28) and $\alpha\beta \equiv 1 \pmod{k}$.*

(i) *A is Hermitian if and only if $P_{\beta\ell}F_{\beta\ell}^* = P_{\alpha\ell}F_\ell, 0 \leqslant \ell \leqslant k-1$.*
(ii) *A is normal if and only if $F_{\beta\ell}F_{\beta\ell}^* = F_\ell^* F_\ell, 0 \leqslant \ell \leqslant k-1$.*
(iii) *A is EP (i.e., $A^\dagger A = AA^\dagger$) if and only if $F_{\beta\ell}F_{\beta\ell}^\dagger = F_\ell^\dagger F_\ell, 0 \leqslant \ell \leqslant k-1$.*

Proof. From (28) and Theorem 5,

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell P_\ell^*, \quad A^* = \sum_{\ell=0}^{k-1} P_\ell F_\ell^* P_{\alpha\ell}^*, \quad \text{and} \quad A^\dagger = \sum_{\ell=0}^{k-1} P_\ell F_\ell^\dagger P_{\alpha\ell}^*. \tag{29}$$

(i) Since $\alpha\beta \equiv 1 \pmod{k}$, replacing ℓ by $\beta\ell$ in the second sum in (29) yields $A^* = \sum_{\ell=0}^{k-1} P_{\beta\ell}F_{\beta\ell}^* P_\ell^*$, and comparing this with the first sum in (29) yields (i).

(ii) From (29),

$$AA^* = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell F_\ell^* P_{\alpha\ell}^* = \sum_{\ell=0}^{k-1} P_\ell F_{\beta\ell}F_{\beta\ell}^* P_\ell^* \quad \text{and} \quad A^*A = \sum_{\ell=0}^{k-1} P_\ell F_\ell^* F_\ell P_\ell^*,$$

which implies (ii).

(iii) From (29),

$$AA^\dagger = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell F_\ell^\dagger P_{\alpha\ell}^* = \sum_{\ell=0}^{k-1} P_\ell F_{\beta\ell}F_{\beta\ell}^\dagger P_\ell^* \quad \text{and} \quad A^\dagger A = \sum_{\ell=0}^{k-1} P_\ell F_\ell^\dagger F_\ell P_\ell^*.$$

which implies (iii). □

Remark 7. If A is a square matrix and there is a matrix B such that $ABA = A$, $BAB = B$, and $AB = BA$, then B is unique and is called the group inverse of A, which is usually denoted by $A^\#$. Davis [6] noted that if $A \in \mathbb{C}^{k\times k}$ is a standard 1-circulant then $A^\dagger = A^\#$. Theorem 10 (iii) extends this: If $A \in \mathbb{C}^{k:d\times d}$ is a proper α-circulant and $\alpha\beta \equiv 1 \pmod{k}$, then $A^\dagger = A^\#$ if and only if $F_\ell^\dagger F_\ell = F_{\beta\ell}F_{\beta\ell}^\dagger, 0 \leqslant \ell \leqslant k-1$.

7. The eigenvalue problem with $\alpha = 1$

In this section we assume that $\alpha = 1$, and $d_1 = d_2 = d$. The following theorem and its proof are motivated by [2, Theorem 2].

Theorem 11. *Let*

$$\mathscr{S}_R = \bigcup_{\ell=0}^{k-1} \{z \mid Rz = \zeta^\ell z\}.$$

If λ is an eigenvalue of A, let $\mathscr{E}_A(\lambda)$ be the λ-eigenspace of A; i.e,

$$\mathscr{E}_A(\lambda) = \{z | Az = \lambda z\}.$$

(i) *If λ is an eigenvalue of $A = [A_{s-r}]_{r,s=0}^{k-1}$ then $\mathscr{E}_A(\lambda)$ has a basis in $\mathscr{S}_R$.*
(ii) *If $A \in \mathbb{C}^{k:d\times d}$ and has kd linearly independent eigenvectors in $\mathscr{S}_R$, then A is a 1-circulant.*

Proof. (i) From Theorem 8, $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell \in \mathscr{E}_A(\lambda)$ if and only if $F_\ell u_\ell = \lambda u_\ell, 0 \leqslant \ell \leqslant k-1$. Therefore λ is an eigenvalue of A if and only if it is an eigenvalue of F_ℓ for some $\ell \in \{0, 1, \dots, k-1\}$. Let $\mathscr{T}_\lambda$ be the subset of $\{0, 1, \dots, k-1\}$ for which this is true. Then $\mathscr{E}_A(\lambda)$ consists of linear combinations of the vectors of the form $P_\ell u_\ell$ with $\ell \in \mathscr{T}_\lambda$ and (λ, u_ℓ) an eigenpair of F_ℓ. Since $RP_\ell = \zeta^\ell P_\ell$ (recall (12)), this completes the proof of (i).

(ii) From Theorem 1, we must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^s z$ then $RAz = \lambda Rz = \lambda\zeta^s z$ and $ARz = \zeta^s Az = \zeta^s \lambda z$. Hence $ARz = RAz$ for all z in a basis for $\mathbb{C}^{k:d\times d}$, so $AR = RA$. $\square$

Theorem 12. *Let R and P be as in (8) and (10). Then the 1-circulant $A = [A_{s-r}]_{r,s=0}^{k-1}$ is diagonalizable if and only if $F_0, F_1, \dots, F_{k-1}$ are all diagonalizable. In this case, if*

$$F_\ell = T_\ell D_\ell T_\ell^*, \quad 0 \leqslant \ell \leqslant k-1$$

are spectral decompositions of $F_0, F_1, \dots, F_{k-1}$ and

$$\Psi = \begin{bmatrix} P_0T_0 & P_1T_1 & \cdots & P_{k-1}T_{k-1} \end{bmatrix},$$

then

$$A = \Psi \left(\bigoplus_{\ell=0}^{k-1} D_\ell \right) \Psi^*$$

is a spectral decomposition of A.

8. The eigenvalue problem with $\alpha > 1$

In this section we assume that $d_1 = d_2 = d, \alpha \in \{2, 3, \dots k-1\}$, and $\gcd(\alpha, k) = 1$. From Theorem 8, $Az = \lambda z$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$, where

$$F_s u_s = \lambda u_{\alpha s}, \quad 0 \leqslant s \leqslant k-1. \tag{30}$$

Therefore $Az = 0$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$ where $F_s u_s = 0$, $0 \leqslant s \leqslant k-1$, so the makeup of the null space of A is transparent. Hence, we assume that $\lambda \neq 0$. Then we must consider the orbits of the permutation on $\{0, \dots, k-1\}$ defined by $s \to \alpha s \pmod k$. We consider an example before presenting the general discussion.

Let $k = 10$ and $\alpha = 3$. The permutation of $\{0, 1, \dots, 9\}$ defined by $s \to 3s \pmod{10}$ is given by

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{pmatrix}.$$

The orbits of this permutation are

$$\mathscr{O}_0 = \{0\}, \quad \mathscr{O}_1 = \{1, 3, 9, 7\}, \quad \mathscr{O}_2 = \{2, 6, 8, 4\}, \quad \text{and} \quad \mathscr{O}_3 = \{5\}.$$

Therefore (30) divides into four independent systems:

$$\text{(i) } F_0u_0 = \lambda u_0; \quad \text{(ii) } F_1u_1 = \lambda u_3, \quad F_3u_3 = \lambda u_9, \quad F_9u_9 = \lambda u_7, \quad F_7u_7 = \lambda u_1,$$
$$\text{(iii) } F_5u_5 = \lambda u_5; \quad \text{(iv) } F_2u_2 = \lambda u_6, \quad F_6u_6 = \lambda u_8, \quad F_8u_8 = \lambda u_4, \quad F_4u_4 = \lambda u_2.$$

From (i), if (λ, u_0) is an eigenpair of F_0 then (λ, P_0u_0) is an eigenpair of A. Similarly, from(iii), if (λ, u_5) is an eigenpair of F_5 then (λ, P_5u_5) is an eigenpair of A. The analysis of (ii) and (iv) is more complicated, but identical. We will consider (ii), which is equivalent to

$$u_3 = \frac{1}{\lambda}F_1 u_1, \quad u_9 = \frac{1}{\lambda}F_3 u_3, \quad u_7 = \frac{1}{\lambda}F_9 u_9, \quad u_1 = \frac{1}{\lambda}F_7 u_7, \tag{31}$$

since $\lambda \neq 0$. Hence,

$$u_3 = \frac{1}{\lambda}G_3 u_1 \quad u_9 = \frac{1}{\lambda^2}G_9 u_1, \quad u_7 = \frac{1}{\lambda^3}G_7 u_1, \quad \text{and} \quad u_1 = \frac{1}{\lambda^4}G_1 u_1, \tag{32}$$

where

$$G_3 = F_1, \quad G_9 = F_3F_1, \quad G_7 = F_9F_3F_1, 7 \quad \text{and} \quad G_1 = F_7F_9F_3F_1. \tag{33}$$

In particular, the last equalities in (32) and (33) are equivalent to $G_1u_1 = \lambda^4 u_1$. Therefore, if (γ, u_1) is an eigenpair of G_1 and $\gamma \neq 0$, then $\lambda = \gamma^{1/4}$ is an eigenvalue of A with the associated eigenvector

$$\begin{aligned} z &= (P_1 + \gamma^{-1/4}P_3G_3 + \gamma^{-2/4}P_9G_9 + \gamma^{-3/4}P_7G_7)u_1 \\ &= \left(P_1 + \sum_{m=1}^{3} \gamma^{-m/4}P_{3^m}G_{3^m}\right)u_1. \end{aligned} \tag{34}$$

(Recall that subscripts are taken modulo 10.) However, $\gamma^{1/4}e^{2\pi ir/4}, 0 \leqslant r \leqslant 3$, are all fourth roots of γ and therefore eigenvalues of A. Replacing $\gamma^{1/4}$ with $\gamma^{1/4}e^{2\pi ir/4}$ in (34) shows that

$$z_r = \left(P_1 + \sum_{m=1}^{3} \gamma^{-m/4}e^{-2\pi irm/4}P_{3^m}G_{3^m}\right)u_1, \quad 0 \leqslant r \leqslant 3, \tag{35}$$

are the respective associated eigenvectors of A.

Now suppose the permutation $s \to \alpha s \pmod{k}$ of $\{0, 1, \ldots, k-1\}$ has p orbits $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{p-1}$, and let

$$0 = s_0 < s_1 < s_2 < \cdots < s_{p-1} \quad \text{with} \quad s_\ell \in \mathcal{O}_\ell, \quad 0 \leqslant \ell \leqslant p-1.$$

Suppose $\mathcal{O}_\ell$ has r_ℓ distinct members; thus,

$$\mathcal{O}_\ell = \{s_\ell, \alpha s_\ell, \ldots, \alpha^{r_\ell-1}s_\ell\}, \quad \text{where} \quad \alpha^{r_\ell} \equiv 1 \pmod{k} \tag{36}$$

and $\bigcup_{\ell=0}^{p-1}\mathcal{O}_\ell = \{0, 1, \ldots, k-1\}$. If $r_\ell = 1$ and (λ, u_{s_ℓ}) is an eigenpair of F_{s_ℓ}, then $(\lambda, P_{s_\ell}u_{s_\ell})$ is an eigenpair of A. Now consider an orbit $\mathcal{O}_\ell$ with $r_\ell > 1$, such as $\mathcal{O}_2$ and $\mathcal{O}_4$ in the example. The system associated with $\mathcal{O}_\ell$ is

$$F_{\alpha^r s_\ell}u_{\alpha^r s_\ell} = \lambda u_{\alpha^{r+1}s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 1, \quad \text{where } \alpha^{r_\ell} = 1,$$

which is analogous to (ii), where $s_\ell = 1, \alpha = 3$ and $k = 10$. Since $\lambda \neq 0$, this is equivalent to

$$u_{\alpha^{r+1}s_\ell} = \frac{1}{\lambda}F_{\alpha^r s_\ell}u_{\alpha^r s_\ell}, \quad 0 \leqslant r_\ell - 1,$$

which is analogous to (31). Therefore

$$u_{\alpha^{r+1}s_\ell} = \frac{1}{\lambda^{r+1}}G_{\alpha^{r+1}s_\ell}u_{s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 1, \tag{37}$$

where

$$G_{\alpha^{r+1}s_\ell} = F_{\alpha^r s_\ell} \cdots F_{s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 1,$$

which is analogous to (32) and (33). In particular, setting $r = r_\ell - 1$ and noting that $\alpha^{r_\ell}s_\ell = s_\ell$ yields

$$u_{s_\ell} = \frac{1}{\lambda^{r_\ell}}G_{s_\ell}u_{s_\ell}, \quad \text{where } G_{s_\ell} = F_{\alpha^{r_\ell-1}s_\ell} \cdots F_{s_\ell}.$$

Therefore, if $(\gamma_\ell, u_{s_\ell})$ is an eigenvalue of G_{s_ℓ}, then γ_ℓ^{1/r_ℓ} is an eigenvalue of A with associated eigenvector

$$z_\ell = \left(P_{s_\ell} + \sum_{m=1}^{r_\ell-1} \gamma_\ell^{-m/r_\ell}P_{\alpha^m s_\ell}G_{\alpha^m s_\ell}\right)u_{s_\ell}, \tag{38}$$

which is analogous to (34). However, since $\gamma^{1/r_\ell}e^{2\pi ir/r_\ell}$ are all r_ℓth roots of γ, they are all eigenvalues of A. Replacing γ^{1/r_ℓ} with $\gamma^{1/r_\ell}e^{2\pi ir/4}$ in (38) yields associated eigenvectors

$$z_{r\ell} = \left(P_{s_\ell} + \sum_{m=1}^{r_\ell-1} \gamma_\ell^{-m/r_\ell} e^{-2\pi irm/r_\ell} P_{\alpha^m s_\ell} G_{\alpha^m s_\ell}\right) u_{s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 1, \tag{39}$$

which is analogous to (35).

Remark 8. Now we apply the preceding argument to a standard α-circulant $A = [a_{s-\alpha r}]_{r,s=0}^{k-1}$ with $\gcd(\alpha, k) = 1$. From Remark 3,

$$A = \sum_{s=0}^{k-1} f_s \phi_{\alpha s}\phi_s^* \quad \text{with} \quad f_s = \sum_{r=0}^{k-1} a_r \zeta^{rs}, \quad 0 \leqslant r \leqslant k-1$$

and $\phi_0, \phi_1, \ldots, \phi_{k-1}$ as in (1). Then $z = \sum_{s=0}^{k-1} u_s\phi_s$ is λ-eigenvector of A if and only if $f_s u_s = \lambda u_{\alpha s}, 0 \leqslant s \leqslant k-1$. Let $\mathcal{O}_\ell$ be as in (36) and assume that $f_{\alpha^r s_\ell} \neq 0, 0 \leqslant r \leqslant r_\ell - 1$. Let

$$g_{\alpha^{r+1}s_\ell} = \prod_{q=0}^{r} f_{\alpha^q s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 1$$

and

$$\gamma_\ell = g_{\alpha^{r_\ell}s_\ell} = f_{\alpha^{r_\ell-1}s_\ell} \cdots f_{s_\ell}.$$

From (37),

$$u_{\alpha^{r+1}s_\ell} = \frac{1}{\lambda^{r+1}} g_{\alpha^{r+1}s_\ell} u_{s_\ell}, \quad 0 \leqslant r \leqslant r_\ell - 2, \quad \text{and} \quad u_{\alpha^{r_\ell}s_\ell} = u_{s_\ell} = \lambda^{-r_\ell}\gamma_\ell u_{s_\ell}.$$

Therefore $\gamma_\ell^{r_\ell} e^{2\pi ir/r_\ell}, 0 \leqslant r \leqslant r_\ell - 1$, are eigenvalues of A. From (39),

$$z_{r\ell} = \left(\phi_{s_\ell} + \sum_{m=1}^{r_\ell-1} \gamma_\ell^{-m/r_\ell} e^{-2\pi irm/r_\ell} g_{\alpha^m s_\ell}\phi_{\alpha^m s_\ell}\right), \quad 0 \leqslant r \leqslant r_\ell - 1,$$

are associated eigenvectors.

For example, let $\alpha = k-1$, so $A = [a_{s+r}]_{r,s=0}^{k-1}$. If $k = 2p$ then $v_0 = f_0, v_\ell = \sqrt{f_\ell f_{k-\ell}}, 1 \leqslant \ell \leqslant p-1$, and $v_p = f_p$. Hence, $(f_0, \phi_0), (f_p, \phi_p)$,

$$\left(\sqrt{f_\ell f_{k-\ell}}, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}}\phi_{(k-1)\ell}\right) \quad \text{and} \quad \left(-\sqrt{f_\ell f_{k-\ell}}, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}}\phi_{(k-1)\ell}\right)$$

$1 \leqslant \ell \leqslant p-1$, are eigenpairs of A. If $k = 2p+1$ then $v_0 = f_0$ and $v_\ell = f_\ell f_{k-\ell}, 1 \leqslant \ell \leqslant q$. Hence (f_0, ϕ_0),

$$\left(\sqrt{f_\ell f_{k-\ell}}, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}}\phi_{(k-1)\ell}\right) \quad \text{and} \quad \left(-\sqrt{f_\ell f_{k-\ell}}, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}}\phi_{(k-1)\ell}\right)$$

$1 \leqslant \ell \leqslant p$, are eigenpairs of A.

The eigenvalues of A were given in [5] without the associated eigenvectors.

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